The proof of Steinberg's three coloring conjecture

I. Cahit

email. icahit@gmail.com

Abstract

The well-known Steinberg's conjecture asserts that any planar graph without 4- and 5-cycles is 3 colorable. In this note we have given a short algorithmic proof of this conjecture based on the spiral chains of planar graphs proposed in the proof of the four color theorem by the author in 2004.

1 Introduction

The problem which we will be dealt with in this note is known as "the three color problem". Clearly the central problem is the four coloring of planar graphs. Ruling out the almost trivial two coloring of bipartite graphs the transition from three coloring to four coloring with respect to graphical property e.g., planarity and non-planarity, cycle sizes, cycles through fix number of vertices, connectivity (disjoint paths between two vertices) of the graph is not very sharp. But there is an old attempt of Heawood for three-color criterion of triangulated planar graphs which has been strengthened in this paper as well as in [1].

Let us briefly list the previous results on the three coloring of planar graphs. Apparently the first 3-color criterion for planar graphs was given by Heawood in 1898 and is known as the "three color theorem"; simply saying that a finite triangulation is 3-colorable if and only if it is even [1]. One may expect that after the proof of four color theorem [2],[3] in 1976 and another earlier result due to Grötzsch saying that "all planar graphs without 3-cycles are three colorable" [4] leads that all coloring problems related with the planar graphs should be easily derived from these results . But Steinberg's three colorability conjecture for planar graphs without 4 and 5 cycles tell us the opposite. Erdös [5] suggested an simpler question by asking whether there exits a constant C such that the absence of cycles with size from 4 to C in a planar graph guarantees its 3-colorability? In response to this question first Abbott and Zhou [6] proved that such a C exists and C≤11. Then Borodin improved it first to 10 and then to C≤9 [7],[8]. Independently Sanders and Zhao have obtained the same upper bound [9]. Very recently Borodin et. al. reduce the upper-bound to 7 by showing that [10]:

Theorem 1 *Every planar graph without cycles of length 4 to 7 is 3-colorable.*

In fact they showed a little stronger result:

Theorem 2 *Every proper 3-coloring of the vertices of any face of size from 8 to 11 in a connected graph G_7 can be extended to a proper 3-coloring of the whole graph.*

In Theorem 2, G_7 denote the class of planar graphs without cycles of size from 4 to 7.

Most of these results on this conjecture are based on the discharging method which was first used in the proof of the four-color theorem. Similarly let us denote by G_6 , the class of planar graphs without cycles of size from 4 to 6. On the other hand in [11] a new 3-color criterion has been given for planar graphs:

Theorem 3 *(3-Color Criteria). Let G be a (biconnected) plane graph. The following three conditions are equivalent:*

(i) G is 3-colorable.
(ii) There exists an even triangulation $H \supseteq G$.
(iii) G is edge-side colorable.

2 Spiral chain coloring algorithm

Our method is not rely on the previous results or necessary conditions of 3 colorability of planar graphs. In fact we have given a coloring algorithm which color vertices of any planar graph in G_6. Let us assume that the G_6 has been embedded in the plane without edge crossing. Since the graph G_6 has no cycle of length four and five this implies that it has also no subgraph such as two triangles with a common edge.

Let c_1, c_2, c_3 be the three colors, say green, yellow, and red. Let us define spiral-chains in G_6 which is exactly same as in [12],[13]. That we select any vertex on the outer-cycle of G_6 and scan outer leftmost vertices in the clockwise (or counter clockwise) direction and bypassing a vertex that is already scanned. If the last scanned vertex is adjacent only previously scanned vertices then we select closest vertex to the last scanned vertex and begin for a new spiral-chain till all vertices scanned. This way we have obtained the set of vertex disjoint spiral chains $S_1, S_2, ..., S_k$ of G_6. Note that if $k = 1$ then S_1 is also an hamilton path of G_6. Coloring of the vertices in $S_1, S_2, ..., S_k$ are carried out in the reverse order of the spiral chains and vertices i.e., $S_k, S_{k-1}, ..., S_1$. A very simple spiral-chain coloring algorithm is given below which is similar the one given for edge coloring of cubic planar graphs in [13] but here we do not need Kempe switching.

Algorithm 4 *Let $S_1, S_2, ..., S_k$ be the spiral chains of G_6 , where spiral chains are ordered from outer-edges towards inner edges.*

Then color respectively the vertices of $S_k, S_{k-1}, ..., S_2, S_1$ using smaller indexed colors whenever possible, that is c_1 (green) is in general the most frequent

and c_3 (red) is the least frequent color used in the algorithm. Note that while coloring the vertices of spiral chain $S_i, k \geq i \geq 1$ if vertices v_j and v_{j+1} of an edge $(v_j, v_{j+1}) \in S_i$ is also belongs to a triangle and has been colored, say with c_1 and c_2 then the other vertex v_k of the triangle will get the color c_3. If v_k has already been colored, say by c_2 then v_j is colored by c_1 and v_{j+1} is colored by c_3.

Cases will be studied in the proof of the following theorem.

Theorem 5 *Algorithm 4 colors the vertices of G_6 with at most three colors.*

Proof. By the ordering of spiral-chains that if $S_i \succ S_j$ then vertices of S_i are colored before the vertices of S_j and also if $v_i \succ v_j$ and $v_i, v_j \in S_k$ then the vertex v_i is colored before v_j. Worst case configuration (a subgraph) in G_6 is a cycle of length six with triangles attached to its edges is shown in Fig.1. The first spiral chain S_1 passing through the three vertices forces all other four spiral chains S_2, S_3, S_4, S_5 ordered in the clockwise direction. Now the spiral-coloring starts from S_5 and go to S_1without needing the forth color as follows:

$S_5 : c_1, c_2, ..., S_4 : c_2, c_3, ..., S_3 : c_1, c_3, ..., S_2 : c_2, c_3, ...,$ and $S_1 : c_2, c_1, c_3, ...$ Note that because of six triangles we have to use the color "red" six times but this is not the case for a less complicated configuration i.e., with less number of triangles, which can be obtained from this one by only deleting vertices colored red.

Consider three disjoint triangles $T_1 = (x_1, y_1, z_1), T_2 = (x_2, y_2, z_2)$ and $T_3 = (x_3, y_3, z_3)$ and three spiral chains $S_i, S_j, S_k, i \leq j \leq k$ respectively passing through the edges $x_i y_i, i = 1, 2, 3$. Let v be a vertex such that $v \in S_p, (z_i, v) \in E(G_6), i = 1, 2, 3$. Without loss of generality assume that $p > k$ that is S_k is coming after S_p and $x_1 y_1 \in S_i, x_2 y_2 \in S_j, x_3 y_3 \in S_k$. We will show that vertices z_1, z_2, z_3 cannot have three different colors but only c_3 by the spiral chain coloring. Assume contrary that the vertices x_i, y_i, z_i have the following coloring:

$x_3 \to c_1, y_3 \to c_2 \Longrightarrow z_3 \to c_2$
$x_2 \to c_2, y_2 \to c_3 \Longrightarrow z_2 \to c_1$
$x_1 \to c_1, y_1 \to c_2 \Longrightarrow z_1 \to c_3.$

But by the coloring rule of the algorithm (piority of the colors in the assignment) z_1, z_2, z_3 are forced to be colored by c_3. So we can start coloring the first vertex v of S_p with $v \to c_1$. Lastly its easy to see that no any vertex of G_6 are left not colored. ■

Based on the algorithm and proof of the above theorem we state.

Theorem 6 *Planar graphs without $4-$ and $5-$cycles are three colorable.*

Example 7 *In Figure 1 we illustrate our spiral-chain three coloring for a graph with 19 vertices in G_6. (a) First we select arbitrarily a vertex from the outer-cycle of the graph and starting from that vertex and traversing the vertices in clockwise direction (always select not yet scanned outer-leftmost vertex) in the form of a spiral chain. Hence the spiral-chain S_1 with vertex set $V(S_1) = \{v_1, v_2, ..., v_{15}\}$. Since the last vertex v_{15} is only adjacent to a vertex of S_1*

we start new spiral chain S_2 from the closest unscanned adjacent vertex (in the graph vertex v_{16}) to v_{15}. Similarly we have obtained the second spiral chain S_2 with the vertex set $V(S_2) = \{v_{16}, v_{17}, v_{18}\}$. Since the other edges connected to v_{18} have adjacent to already scanned spiral chain S_1 vertices (v_{10} and v_{11}) we start a new spiral-chain S_3 closest unscanned vertex which is a isolated vertex $V(S_3) = \{v_{19}\}$. (b) Second part of the algorithm is the three-coloring of the vertices of the spiral-chains S_3, S_2, S_1 in reveres order. Colors c_1 (green),c_2 (yellow) and c_3 (red) have been denoted by the numbers $1, 2$ and 3. Note that in the spiral-chain coloring algorithm the color "green" has a pioroty over "yellow" and "red" colors and the color yellow has a piority over the red color. Color the vertex in $V(S_3) = \{v_{19}\}$ with $v_{19} \to c_1$ and since (v_{19}, v_{14}, v_{13}) is a triangle we also color $v_{14} \to c_2, v_{13} \to c_3$. Color the vertices of $V(S_2) = \{v_{18}, v_{17}, v_{16}\}$ as $v_{18} \to c_1, v_{17} \to c_2, v_{16} \to c_1$. Since (v_{18}, v_{11}, v_{10}) is a triangle we also color $v_{12} \to c_2, v_{10} \to c_3$. Similarly since (v_{16}, v_8, v_7) is a triangle we color $v_8 \to c_2$ and $v_7 \to c_3$. Finally color the vertices of $V(S_1) = \{v_{15}, v_{14}, ..., v_1\}$ as $v_{15} \to c_1$, since (v_{15}, v_7, v_6) is a triangle color also $v_7 \to c_3, v_7 \to c_2$.Since v_2 and v_3 have been already colored with c_2 and c_3 we continiue coloring $v_{12} \to c_1$ and since v_{11} and v_{10} have been colored before with c_2 and c_3 we color $v_9 \to c_1$.Since v_8, v_7 and v_6 are colored before with c_2, c_3 and c_2 we color $v_6 \to c_1, v_3 \to c_1$ and $v_1 \to c_1$.

Acknowledgement 8 *I like to thank to C. C. Heckman for his interest and views on this problem.*

References

[1] P. J. Heawood, "On the four-color map theorem", Quart. J. Pure Math. 29 (1898) 270-285.

[2] K. Appel and W. Haken, "Every planar map is four colorable", Contemporary Math., 98 (1989).

[3] N. Robertson, D. P. Sanders, P. D. Seymour and R. Thomas, "The four colour theorem", J. Combin. Theory Ser. B. 70 (1997), 2-44.

[4] H. Groetzsch, "Ein Dreifarbensatz fur Dreikreisfrei Netze auf der Kugel", Wiss. Z. Martin Luther Univ. Halle-Wittenburg Math. Naturwiss. Reihe, vol 8, 1958.

[5] R. Steinberg, "The state of the three color problem", Quo Vadis, Graph Theory? J. Gimbel, J.W. Kennedy and L.V. Quintas (eds), Ann. Disc. Math., 55 (1993) 211-248.

[6] H.L. Abbott and B. Zhou, "On small faces in 4-critical graphs", Ars Combin., 32 (1991) 203-207.

[7] O.V. Borodin, "To the paper of H.L. Abbott and B. Zhou on 4-critical planar graphs", Ars Combin., 25 (1979) 211-236.

[8] O.V. Borodin " Structural properties of plane graphs without adjacent triangles and an application to 3-coloring", J. of Graph Theory, 21 (2) (1996) 183-186.

[9] D.P. Sanders and Y. Zhao, " A note on the Three Color Problem", Graphs and Combinatorics, 11 (1995) 91-94.

[10] O.V. Borodin, A.N. Glebov, A. Raspaud and M.R. Salavatipour, "Planar graphs without cycles of length 4 to 7 are 3-colorable", J. of Combinatorial Theory (Series B), 93: 303-311, 2005.

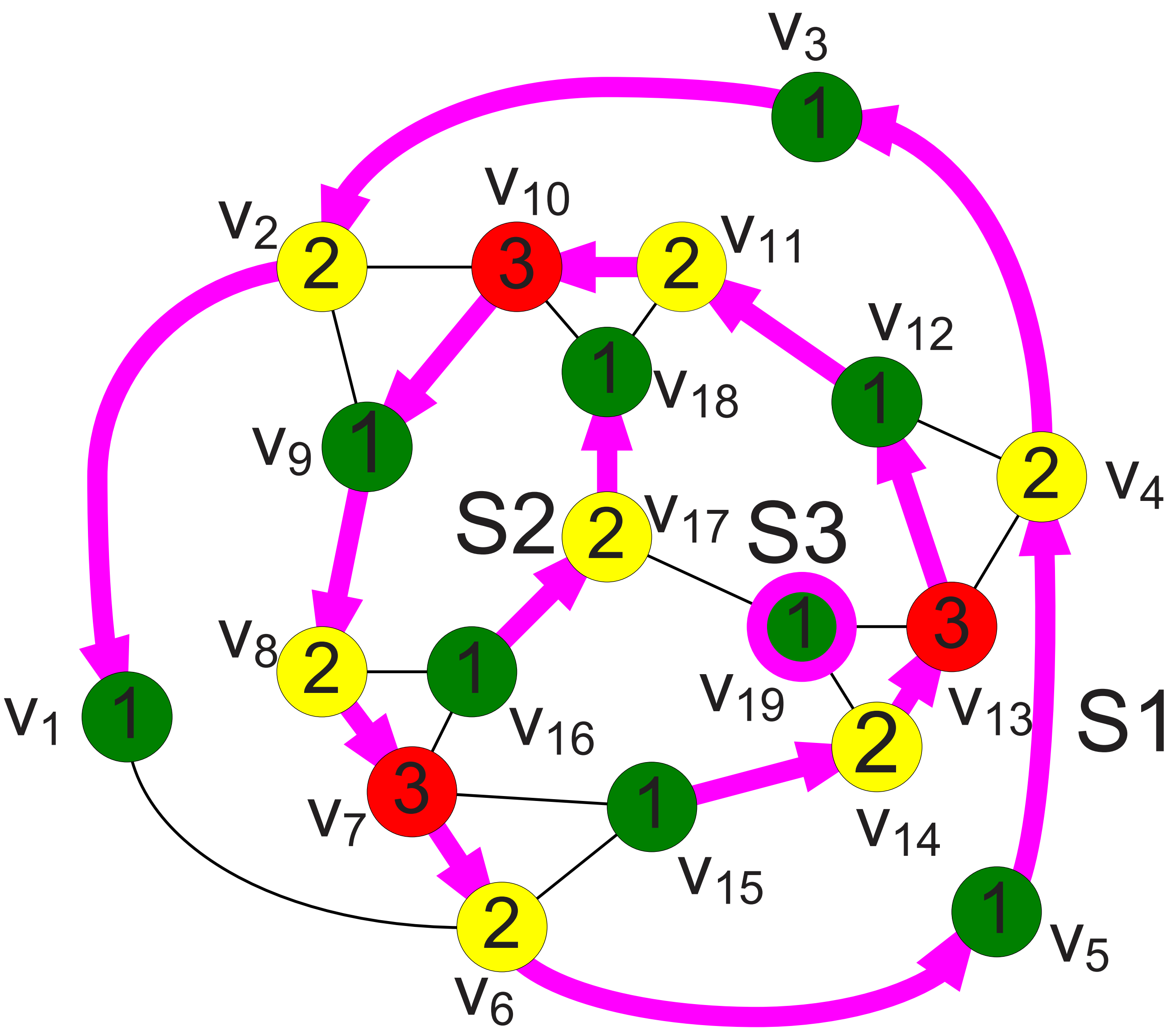
[11] K. Diks, L. Kowalik and M. Kurowski, "A new 3-color criterion for planar graphs", WG2002, LNCS 2573, L. Kucera (ed.), (2002) 138-149, Springer-Verlag Berlin Heidelberg 2002.

[12] I. Cahit, "Spiral chains: A new proof of the four color theorem", arXiv preprint, `math CO/0408247`, August 18, 2004.

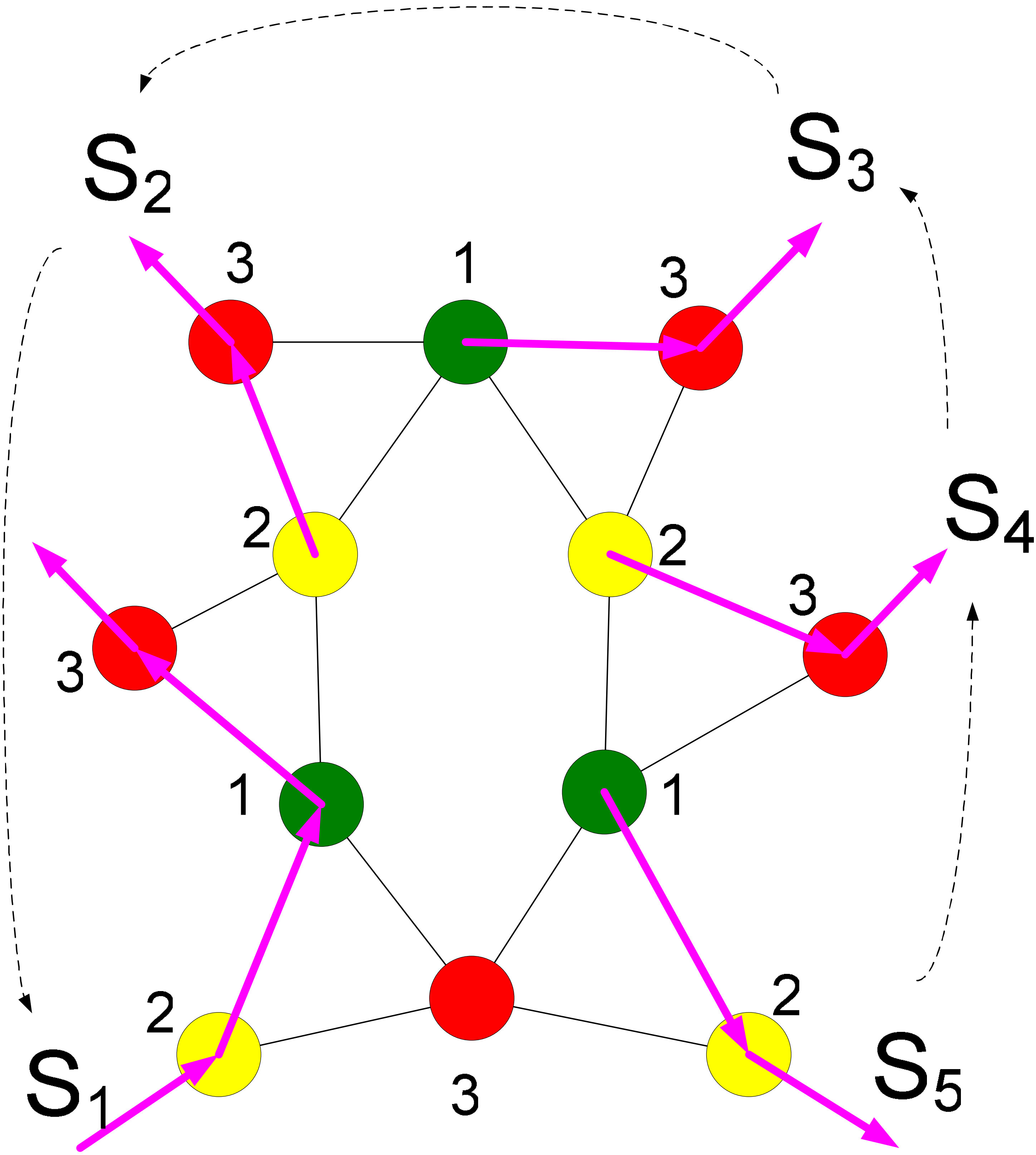
[13] I. Cahit, "Spiral chains: The proofs of Tait's and Tutte's three edge-coloring conjectures", arXiv prepint, `math CO/0507127 v1`, July 6, 2005.

Figure 1: Worst case three coloring around cycle of length three. Coloring is carried out from high index spiral chains toward low index spiral chains.

Figure 2: Illustration of the spiral-chain three coloring.



v3
1
v2
2
v10
3
v11
2
v12
1
v18
1
v9
1
v4
2
S2
2
v17
S3
1
3
v8
2
1
v19
v13
S1
v1
1
v16
2
3
1
v7
v14
v15
1
v5
2
v6



S2
S3
3
1
3
2
2
S4
3
3
1
1
2
2
S1
3
S5